Bull's Eye Solution

Let's look at just one side of the target with the center of the target at the origin. We'll assume for simplicity the target's edge is 1 unit from the center.

We want the boundary to be set up so a point on the boundary is \( r \) units from both the center and the edge of the target as shown. We'll give a polar function relating \( \theta \) and \( r \).

\[
r(\theta) = (1-r(\theta)) \sec \theta \quad \text{soluling for } r(\theta)
\]

we see \( r(\theta) = \frac{\sec \theta}{1+\sec \theta} \)

so \( r(\theta) = \frac{\sec \theta \cdot \cos \theta}{1+\sec \theta \cdot \cos \theta} = \frac{1}{1+\cos \theta} \quad \text{and this is valid for } \theta \in \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right] \)
To find the area we must integrate

\[ \frac{3}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \left( \frac{1}{1 + \cos \theta} \right)^2 \, d\theta \]

First a bit more trig. substitutions:

\[ \frac{1 + \cos \theta}{2} = \cos^2 \frac{\theta}{2} \implies 2 \cos^2 \frac{\theta}{2} = 1 + \cos \theta \]

So,

\[ \frac{3}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{4 \cos^4 \frac{\theta}{2}} \, d\theta \text{ is what we want.} \]

This equals

\[ \frac{3}{8} \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^4 \frac{\theta}{2} \, d\theta = \frac{3}{4} \int_{\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2 \phi \, d\phi \]

\[ = \frac{3}{4} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2 \phi \left( 1 + \tan^2 \phi \right) \, d\phi = \frac{3}{4} \tan \frac{\pi}{2} - \frac{3}{4} \tan \frac{-\pi}{2} \]

\[ = \frac{3}{4} \left( \tan \frac{\pi}{2} + \frac{1}{3} \tan \frac{\pi}{3} \right) = \frac{3}{4} \left[ \frac{\sqrt{3}}{3} + \frac{1}{9\sqrt{3}} \right] - \left( \frac{1}{\sqrt{3}} + \frac{1}{9\sqrt{3}} \right) \]

\[ = \frac{60}{36\sqrt{3}} = \frac{5}{3\sqrt{3}}. \]
The area of the entire target is much easier to calculate.

\[ \text{by the pythagorean theorem we get that } 2 \sqrt{3} \text{ is the length of one side of the triangle making the area } 3 \sqrt{3} \]

The ratio of areas then is

\[ \frac{5}{3\sqrt{3}} : 3\sqrt{3} \text{ giving } \frac{5}{\sqrt{27}} \]